Consistency between coordinate transformation rules and dual basis concepts.

We defined $df$, $f$ $C^\infty$ function, by $df(v) = vf$ for each $p \in M$, $v \in T_pM$. Separately, we defined, given a local coordinate system $(x_1, \ldots, x_n)$, the items $dx_1, \ldots, dx_n$ at $p \in M$ to be the dual basis in $(T_pM)^*$ ($=$ dual of $T_pM$), dual to $\frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_n}\big|_p$. These definitions are consistent because $dx_i\left(\frac{\partial}{\partial x_j}\right)$ at $p$ is, according to the df definition, $= \frac{\partial}{\partial x_j}(dx_i)$ at $p$ which $=$ 1 if $j=i$, 0 if $j \neq i$, consistently with the the definition of the basis dual to $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$.

It is interesting to see how this all works under coordinate change. The definition of $df$ does not involve coordinate choice as such. But it is easy to see that in local coordinates

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Proof: If $v = \sum a_j \frac{\partial}{\partial x_j}$ at $p$, then $df(v) \stackrel{\text{def}}{=} vf$
$= \sum a_j \frac{\partial f}{\partial x_j}\big|_p = \sum \frac{\partial f}{\partial x_j} dx_j(v) = \left(\sum \frac{\partial f}{\partial x_j} dx_j\right) v$

since $dx_j(v) = a_j$ (because $\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\right) = 1$ if $i=j$, 0 if $i \neq j$)
So $df$ and $\sum \frac{\partial f}{\partial x_i} dx_i$ give the same answer for all $v \in T_pM$ and hence are equal as elements of $(T_pM)^*$.

Now we have the formula, if $(\hat{x}_1, \dots \hat{x}_n)$ are also loc. coords: $d\hat{x}_i = \sum_j \frac{\partial \hat{x}_i}{\partial x_j} dx_j$ and we can

check that this formula is consistent with $d\hat{x}_1, \dots, d\hat{x}_n$ being the $(T_pM)^*$ basis dual to $\frac{\partial}{\partial \hat{x}_1}\Big|_p, \dots \frac{\partial}{\partial \hat{x}_n}\Big|_p$ basis for $T_pM$: namely we used $\frac{\partial}{\partial \hat{x}_\ell} = \sum_k \frac{\partial x_k}{\partial \hat{x}_\ell} \frac{\partial}{\partial x_k}$ so that

$$\sum_j \left(\frac{\partial \hat{x}_i}{\partial x_j} dx_j\right)\left(\frac{\partial}{\partial \hat{x}_\ell}\right) = \sum_{j,k} \frac{\partial \hat{x}_i}{\partial x_j} \cdot \frac{\partial x_k}{\partial \hat{x}_\ell} dx_j\left(\frac{\partial}{\partial x_k}\right)$$

$$= \sum_j \frac{\partial \hat{x}_i}{\partial x_j} \frac{\partial x_j}{\partial \hat{x}_\ell} \qquad \text{since } dx_j\left(\frac{\partial}{\partial x_k}\right) = 1 \text{ if } j=k,\ 0 \text{ if } j \neq k$$

But $\sum_j \frac{\partial \hat{x}_i}{\partial x_j} \frac{\partial x_j}{\partial \hat{x}_\ell} = 1$ if $i=\ell$, $0$ if $i \neq \ell$

since $\left(\frac{\partial \hat{x}_i}{\partial x_j}\right)$ and $\left(\frac{\partial x_j}{\partial \hat{x}_\ell}\right)$ are inverse matrices of each other. Thus $\sum \frac{\partial \hat{x}_i}{\partial x_j} dx_j$ $i=1, \dots n$ really is the dual basis of $(T_pM)^*$, dual to $\frac{\partial}{\partial \hat{x}_1}, \dots, \frac{\partial}{\partial \hat{x}_n}$.

We can illustrate this (as usual) by polar coordinates and rectangular coordinates on $\mathbb{R}^2$.

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy = \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy$$

(from $r = \sqrt{x^2+y^2}$) and $d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy$

$$= -\frac{y}{x^2+y^2} \frac{\partial}{\partial x} + \frac{x}{x^2+y^2} \frac{\partial}{\partial y} \quad \left(\text{from } \theta = \arctan\left(\frac{y}{x}\right)\right)$$

From before $\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$

while $\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$.

So

$$dr\left(\frac{\partial}{\partial r}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy\right)\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}}\cos\theta\ dx\left(\frac{\partial}{\partial x}\right) + \frac{y}{\sqrt{x^2+y^2}}\sin\theta\, dy\left(\frac{\partial}{\partial y}\right)$$

(no "cross terms" since $dx\left(\frac{\partial}{\partial y}\right)=0$ and $dy\left(\frac{\partial}{\partial x}\right)=0$)

$$= \frac{r\cos\theta}{r}\cos\theta\cdot 1 + \frac{r\sin\theta}{r}\sin\theta$$

$$= \cos^2\theta + \sin^2\theta = 1$$

while

$$dr\left(\frac{\partial}{\partial\theta}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy\right)\left(-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}}(-r\sin\theta) + \frac{y}{\sqrt{x^2+y^2}}(r\cos\theta)$$

$$= \frac{r\cos\theta}{r}(-r\sin\theta) + \frac{r\sin\theta}{r}(r\cos\theta)$$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0.$$

You can try $d\theta\left(\frac{\partial}{\partial r}\right)=0$ and $d\theta\left(\frac{\partial}{\partial\theta}\right)=1$ for yourself!
But actually $d\theta\left(\frac{\partial}{\partial\theta}\right)=1$ is interesting enough not to leave as an exercise:

$$d\theta\left(\frac{\partial}{\partial\theta}\right) = \left(-\frac{r\sin\theta}{r^2}dx + \frac{r\cos\theta}{r^2}dy\right)\left(-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right)$$

$$= -\frac{r\sin\theta}{r^2}\cdot -r\sin\theta + \frac{r\cos\theta}{r^2}\, r\cos\theta$$

$$= \sin^2\theta + \cos^2\theta = 1.$$ This all clearly encodes something interesting.